FP2 2003

1. (i) (*a*) On the same Argand diagram sketch the loci given by the following equations.

$$|z-1| = 1, \quad ,, \arg(z+1) = \frac{\pi}{12}, \quad , \arg(z+1) = \frac{\pi}{2}. \qquad \textbf{(4)}$$

(*b*) Shade on your diagram the region for which

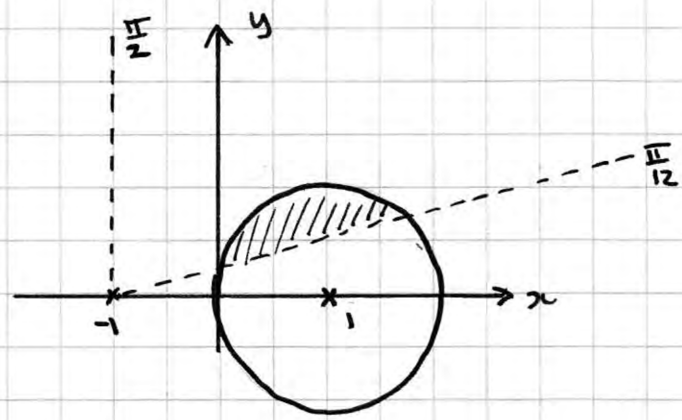
$$|z-1| \le 1 \quad \text{and} \quad \frac{\pi}{12} \le \arg(z+1) \le \frac{\pi}{2}. \qquad \textbf{(1)}$$

(ii) (*a*) Show that the transformation $\quad w = \dfrac{z-1}{z}, \quad z \ne 0,$

maps $|z-1| = 1$ in the *z*-plane onto $|w| = |w-1|$ in the *w*-plane. **(3)**

The region $|z-1| \le 1$ in the *z*-plane is mapped onto the region *T* in the *w*-plane.

(*b*) Shade the region *T* on an Argand diagram. **(2)**



$wz = z - 1$

$1 = z - wz$

$1 = z(1-w)$

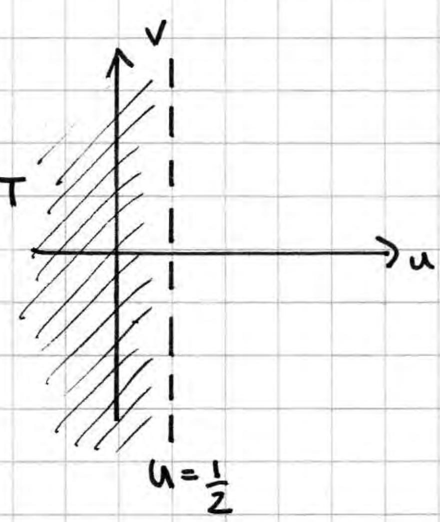
$z = \dfrac{1}{1-w}$

$z - 1 = \dfrac{1}{1-w} - \dfrac{1(1-w)}{(1-w)}$

$\Rightarrow z - 1 = \dfrac{w}{1-w} \Rightarrow |z-1| = \dfrac{|w|}{|1-w|} \Rightarrow 1 = \dfrac{|w|}{|1-w|} \therefore |w| = |1-w|$

$\Rightarrow |w| = |w-1|$ #

b) $|z-1| \le 1 \Rightarrow \dfrac{|w|}{|w-1|} \le 1 \Rightarrow |w| \le |w-1|$



2. (a) Use de Moivre's theorem to show that

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$

(6)

(b) Hence find 3 distinct solutions of the equation $16x^5 - 20x^3 + 5x + 1 = 0$, giving your answers to 3 decimal places where appropriate.

(4)

$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

$(\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$

equating real parts $\Rightarrow$

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1 - 2\cos^2\theta + \cos^4\theta)$

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$

$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ #

$16x^5 - 20x^3 + 5x = -1 \Rightarrow \cos 5\theta = -1$ if $x = \cos\theta$

$5\theta = \cos^{-1}(-1) = \pi, 3\pi, 5\pi \Rightarrow \theta = \frac{\pi}{5}, \frac{3\pi}{5}, \pi$

$\therefore x = \cos\frac{\pi}{5} = 0.809$

$x = \cos\frac{3\pi}{5} = -0.309$

$x = \cos\pi = -1$

3. $$\frac{dy}{dx} = x^2 - y^2, \quad y = 1 \text{ at } x = 0. \quad \text{(I)}$$

(*b*) By differentiating (I) twice with respect to *x*, show that

$$\frac{d^3y}{dx^3} + 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 - 2 = 0. \text{ (4)}$$

(*c*) Hence, for (I), find the series solution for *y* in ascending powers of *x* up to and including the term in x^3. (4)

$$\frac{d^2y}{dx^2} = 2x - 2y\frac{dy}{dx}$$

$$\frac{d^3y}{dx^3} = 2 - 2\left(\frac{dy}{dx}\right)^2 - 2y\frac{d^2y}{dx^2} \qquad \therefore \frac{d^3y}{dx^3} + 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 - 2 = 0 \quad \#$$

$x_0 = 0 \quad y_0 = 1 \qquad y'_0 = (0)^2 - (1)^2 = -1$

$y''_0 = 2(0) - 2(1)(-1) = 2$

$y'''_0 = 2 - 2(-1)^2 - 2(1)(2) = -4$

$$\therefore \quad y = 1 - x + x^2 - \frac{2}{3}x^3$$

2

4. (*a*) Express as a simplified single fraction $\dfrac{1}{(r-1)^2} - \dfrac{1}{r^2}$. **(2)**

(*b*) Hence prove, by the method of differences, that $\displaystyle\sum_{r=2}^{n} \frac{2r-1}{r^2(r-1)^2} = 1 - \frac{1}{n^2}$. **(3)**

a) $\dfrac{1}{(r-1)^2} - \dfrac{1}{r^2} = \dfrac{r^2-(r-1)^2}{r^2(r-1)^2} = \dfrac{2r-1}{r^2(r-1)^2}$

b) $\displaystyle\sum_{r=2}^{n} \frac{2r-1}{r^2(r-1)^2} = \underbrace{\left(\frac{1}{1} - \cancel{\frac{1}{4}}\right)}_{r=2} + \underbrace{\left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{9}}\right)}_{r=3} + \underbrace{\left(\cancel{\frac{1}{9}} - \cancel{\frac{1}{16}}\right)}_{r=4} + \dots + \overbrace{\left(\cancel{\frac{1}{(n-2)^2}} - \cancel{\frac{1}{(n-1)^2}}\right)}^{r=n-1} + \overbrace{\left(\cancel{\frac{1}{(n-1)^2}} - \frac{1}{n^2}\right)}^{r=n}$

$\therefore \; = 1 - \dfrac{1}{n^2}$ #

Solve the inequality $\frac{1}{2x+1} > \frac{x}{3x-2}$.

(6)

$$\frac{(2x+1)^{\cancel{2}}(3x-2)^2}{(\cancel{2x+1})} > \frac{x(2x+1)^2(3x-2)^{\cancel{2}}}{(\cancel{3x-2})}$$

$$(2x+1)(3x-2)^2 - x(2x+1)^2(3x-2) > 0$$

$$(2x+1)(3x-2)[(3x-2) - x(2x+1)] > 0$$

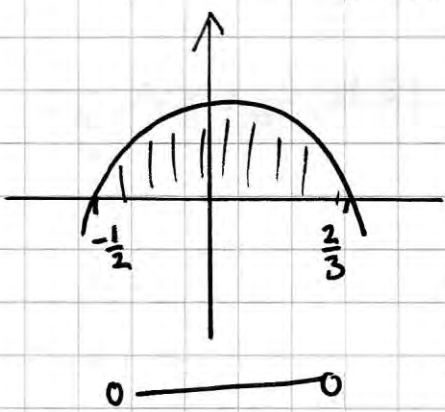
$$(2x+1)(3x-2)[3x-2-2x^2-x] > 0$$

$$-(2x+1)(3x-2)(2x^2-2x+2) > 0$$

$$-(2x+1)(3x-2)(2x \quad)(x- \quad) > 0$$

$b^2-4ac = -12 \therefore$ no solution

always >0



$\therefore \quad -\frac{1}{2} < x < \frac{2}{3}$

6. (*a*) Using the substitution $t = x^2$, or otherwise, find

$$\int x^3 e^{-x^2}\ dx.$$

(6)

(*b*) Find the general solution of the differential equation

$$x\frac{dy}{dx} + 3y = xe^{-x^2}, \quad x > 0.$$

(4)

$$\int x^3 e^{-x^2} dx \qquad t = x^2$$

$$\frac{dt}{dx} = 2x \qquad \frac{1}{2} dt = x\,dx$$

$$\int x^2 e^{-x^2} x\,dx \Rightarrow \frac{1}{2}\int t e^{-t} dt$$

$u = \frac{1}{2}t \qquad v = -e^{-t}$

$u' = \frac{1}{2} \qquad v' = e^{-t}$

$$= -\frac{1}{2}te^{-t} + \frac{1}{2}\int e^{-t} dt$$

$$= -\frac{1}{2}te^{-t} - \frac{1}{2}e^{-t} = -\frac{1}{2}e^{-t}(t+1) \Rightarrow -\frac{1}{2}e^{-x^2}(x^2+1)$$

b) $\frac{dy}{dx} + \frac{3}{x}y = e^{-x^2} \qquad \text{IF } f(x) = e^{\int \frac{3}{x}dx} = (e^{\ln x})^3 = x^3$

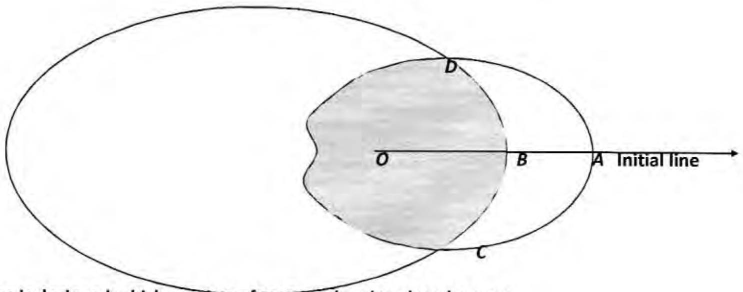
$$\Rightarrow x^3\frac{dy}{dx} + 3x^2y = x^3e^{-x^2} \qquad \Rightarrow \frac{d}{dx}(x^3y) = x^3e^{-x^2}$$

$$\Rightarrow x^3y = \int x^3e^{-x^2} = -\frac{1}{2}e^{-x^2}(x^2+1) + c$$

$$\therefore\ y = \frac{c - \frac{1}{2}e^{-x^2}(x^2+1)}{x^3}$$

7.

Figure 1



A logo is designed which consists of two overlapping closed curves.

The polar equations of these curves are $r = a(3 + 2\cos\theta)$ and

$$r = a(5 - 2\cos\theta), \quad 0 \le \theta < 2\pi.$$

Figure 1 is a sketch (not to scale) of these two curves.

(*a*) Write down the polar corrdinates of the points *A* and *B* where the curves meet the initial line.(2)

(*b*) Find the polar coordinates of the points *C* and *D* where the two curves meet. (4)

(*c*) Show that the area of the overlapping region, which is shaded in the figure, is

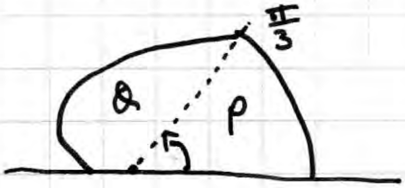
$$\frac{a^2}{3}(49\pi - 48\sqrt{3}) \qquad (8)$$

$A(5a, 0)$

$B(3a, 0)$

$\cancel{a}(3+2\cos\theta) = \cancel{a}(5-2\cos\theta) \Rightarrow 4\cos\theta = 2 \Rightarrow \cos\theta = \frac{1}{2} \quad \therefore \theta = \frac{\pi}{3}, -\frac{\pi}{3}$

$\therefore \; D(4a, \frac{\pi}{3}) \quad C(4a, \frac{5\pi}{3}) \qquad r = a(3+2(\frac{1}{2})) = 4a$



Area $= 2(P+Q)$

$$\text{Area} = \cancel{2}\left[\frac{1}{\cancel{2}}\int_0^{\frac{\pi}{3}} a^2(5-2\cos\theta)^2\,d\theta + \frac{1}{\cancel{2}}\int_{\frac{\pi}{3}}^{\pi} a^2(3+2\cos\theta)^2\,d\theta\right]$$

$$= a^2\left[\int_0^{\frac{\pi}{3}} 25 - 20\cos\theta + 4(\tfrac{1}{2}+\tfrac{1}{2}\cos2\theta) + \int_{\frac{\pi}{3}}^{\pi} 9 + 12\cos\theta + 4(\tfrac{1}{2}+\tfrac{1}{2}\cos2\theta)\,d\theta\right]$$

$$= a^2\left[\int_0^{\pi/3} 27 - 20\cos\theta + 2\cos2\theta\,d\theta + \int_{\pi/3}^{\pi} 11 + 12\cos\theta + 2\cos2\theta\,d\theta\right]$$

$$= a^2\left[\left[27\theta - 20\sin\theta + \sin2\theta\right]_0^{\pi/3} + \left[11\theta + 12\sin\theta + \sin2\theta\right]_{\frac{\pi}{3}}^{\pi}\right]$$

$$= a^2\left[\left[9\pi - 10\sqrt{3} + \frac{\sqrt{3}}{2}\right] + \left[(11\pi) - \left(\frac{11\pi}{3} + 6\sqrt{3} + \frac{\sqrt{3}}{2}\right)\right]\right]$$

$$= a^2\left[\frac{49\pi}{3} - 16\sqrt{3}\right] = \frac{1}{3}a^2(49\pi - 48\sqrt{3})$$

8. $$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = 4e^{3t}, \quad t \geq 0.$$

(*a*) Show that Kt^2e^{3t} is a particular integral of the differential equation, where K is a constant to be found. (4)

(*b*) Find the general solution of the differential equation. (3)

Given that a particular solution satisfies $y = 3$ and $\frac{dy}{dt} = 1$ when $t = 0$,

(*c*) find this solution.(4)

Another particular solution which satisfies $y = 1$ and $\frac{dy}{dt} = 0$ when $t = 0$, has equation

$$y = (1 - 3t + 2t^2)e^{3t}.$$

(*d*) For this particular solution draw a sketch graph of y against t, showing where the graph crosses the t-axis. Determine also the coordinates of the minimum of the point on the sketch graph.

a) $y = kt^2e^{3t}$

$y' = 2kte^{3t} + 3kt^2e^{3t} = (2kt + 3kt^2)e^{3t}$

$y'' = (2k + 6kt)e^{3t} + 3(2kt + 3kt^2)e^{3t}$

$y'' = (2k + 12kt + 9kt^2)e^{3t}$

$$\begin{array}{ll} y'' & = (2k + 12kt + 9kt^2)e^{3t} \\ -6y' & \quad (-12kt - 18kt^2)e^{3t} \\ +9y & \quad + 9kt^2e^{3t} \\ \hline 4e^{3t} & = 2ke^{3t} \quad \therefore \underline{k = 2} \end{array}$$

$\underline{\underline{y_{PI} = 2t^2e^{3t}}}$

$y = Ae^{mt}$ $\quad y'' - 6y' + 9y = 0$ $\quad y_{cf} = (A + Bt)e^{3t}$

$y' = Ame^{mt}$ $\quad Ae^{mt}(m^2 - 6m + 9) = 0$

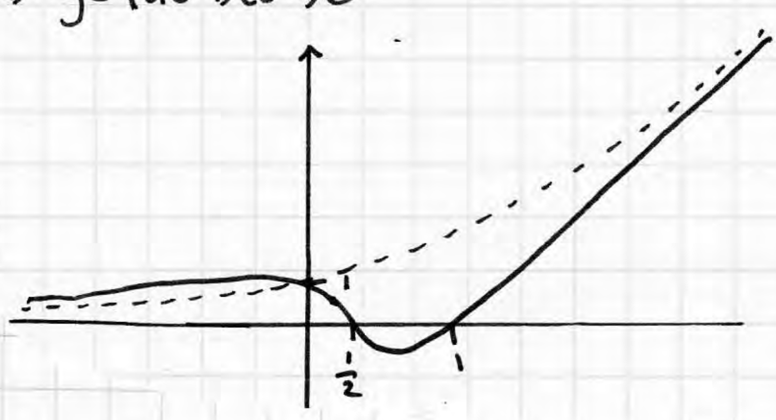
$y'' = Am^2e^{mt}$ $\quad \neq 0 \quad (m-3)^2 = 0 \quad \therefore m = 3$ RR

b) $y = (A + Bt + 2t^2)e^{3t}$

c) $t=0, y=1 \Rightarrow \underline{1 = A}$ $\quad y = (1 + Bt + 2t^2)e^{3t} \Rightarrow y' = (B + 4t)e^{3t} + 3(1 + Bt + 2t^2)e^{3t}$

$t=0, y'=0 \Rightarrow 0 = B + 3 \therefore \underline{B = -3}$ $\quad \therefore \underline{\underline{y = (1 - 3t + 2t^2)e^{3t}}}$

d) $y = (2t-1)(t-1)e^{3t}$



as $t \to \infty$ $y \to e^{3t}$

as $t \to -\infty$ $y \to 0$

$y = 0$ at $\frac{1}{2}, 1$

$t = 0$ $y = e^{3\times 0} = 1$

$y' = (4t - 3)e^{3t} + 3(1 - 3t + 2t^2)e^{3t}$

$y' = (6t^2 - 5t)e^{3t} = t(6t - 5)e^{3t}$

$y' = 0$ at TP $\quad$ TP $t = 0$ $t = \frac{5}{6}$

9. $z = 4\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)$, and $w = 3\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right)$.

Express zw in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, $r > 0$, $-\pi < \theta < \pi$.

(3)

$$zw = 12\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

$$= 12\left(\cos\frac{\pi}{4}\cos\frac{2\pi}{3} - \sin\frac{\pi}{4}\sin\frac{2\pi}{3}\right) + i\left(\cos\frac{2\pi}{3}\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\sin\frac{2\pi}{3}\right)$$

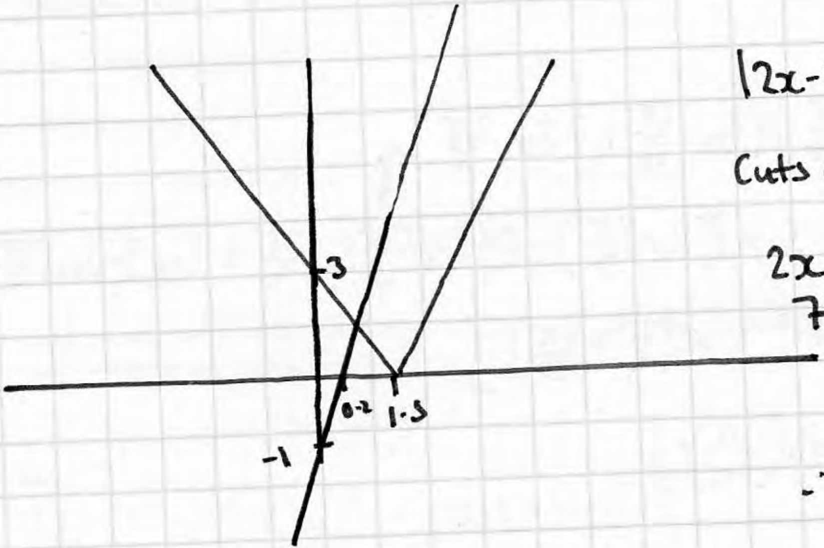
$$= 12\left[\cos\left(\frac{\pi}{4} + \frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{4} + \frac{2\pi}{3}\right)\right]$$

$$= 12\left[\cos\left(\frac{11}{4}\pi\right) + i\sin\left(\frac{11\pi}{4}\right)\right]$$

10. (*a*) Sketch, on the same axes, the graphs with equation $y = |2x - 3|$, and the line with equation $y = 5x - 1$.

(2)

(*b*) Solve the inequality $|2x - 3| < 5x - 1$. (3)



$|2x-3| = 5x-1$

Cuts at reflected part of

$2x - 3 = 1 - 5x$

$7x = 4$

$x = \frac{4}{7}$

$\therefore \ x > \frac{4}{7}$

11. (*a*) Express $\dfrac{2}{(r+1)(r+3)}$ in partial fractions. (2)

(*b*) Hence prove that $\displaystyle\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)} \equiv \frac{n(5n+13)}{6(n+2)(n+3)}$. (5)

$$\frac{2}{(r+1)(r+3)} = \frac{A}{r+1} + \frac{B}{r+3} \Rightarrow 2 = A(r+3) + B(r+1)$$

$r=-1 \Rightarrow A=1$

$r=-3 \Rightarrow B=-1$

$$\frac{1}{r+1} - \frac{1}{r+3}$$

b) $\displaystyle\sum_{1}^{n} \frac{2}{(r+1)(r+3)} = \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) +$ $\quad r=1$

$\left(\frac{1}{3} - \cancel{\frac{1}{5}}\right) +$ $\quad r=2$

$\left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) + \dots +$ $\quad r=3$

$\left(\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n+1}}\right) +$ $\quad r=n-2$

$\left(\cancel{\frac{1}{n}} - \frac{1}{n+2}\right) +$ $\quad r=n-1$

$\left(\cancel{\frac{1}{n+1}} - \frac{1}{n+3}\right)$ $\quad r=n$

$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}$$

$$= \frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3}$$

$$= \frac{5(n+2)(n+3) - 6(n+3) - 6(n+2)}{6(n+2)(n+3)}$$

$$= \frac{5n^2 + 25n + \cancel{30} - 6n - \cancel{18} - 6n - \cancel{12}}{6(n+2)(n+3)} = \frac{n(5n+13)}{6(n+2)(n+3)}$$

12. (*a*) Use the substitution $y = vx$ to transform the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(4x+y)(x+y)}{x^2}, \ x > 0 \qquad \text{(I)}$$

into the equation $$x\frac{\mathrm{d}v}{\mathrm{d}x} = (2+v)^2. \qquad \text{(II)} \qquad \text{(4)}$$

(*b*) Solve the differential equation II to find v as a function of x (5)

(*c*) Hence show that $y = -2x - \dfrac{x}{\ln x + c}$, where c is an arbitrary constant, is a general solution of the differential equation I.

(1)

$$y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{(4x+vx)(x+vx)}{x^2} = \frac{\cancel{x^2}(4+v)(1+v)}{\cancel{x^2}}$$

$$\Rightarrow x\frac{dv}{dx} = (4+v)(1+v) - v = v^2 + 4v + 4$$

$$\therefore x\frac{dv}{dx} = (v+2)^2 \quad \text{II}$$

b) $$\Rightarrow \int (v+2)^{-2}\,dv = \int \frac{1}{x}\,dx \Rightarrow -(v+2)^{-1} = \ln x + c$$

$$\Rightarrow \frac{1}{v+2} = -\ln x - c = \ln x^{-1} + d$$

$$\Rightarrow v + 2 = \frac{1}{A - \ln x} \qquad \therefore v = \frac{1}{A - \ln x} - 2$$

c) $$\frac{y}{x} = -2 - \frac{1}{\ln x + f} \qquad \therefore y = -2x - \frac{x}{\ln x + f}$$

13. Given that $z = 3 - 3\mathrm{i}$ express, in the form $a + \mathrm{i}b$, where a and b are real numbers,

(*a*) z^2, (2) (*b*) $\frac{1}{z}$. (2)

(*c*) Find the exact value of each of $|z|$, $|z^2|$ and $\left|\frac{1}{z}\right|$. (2)

The complex numbers z, z^2 and $\frac{1}{z}$ are represented by the points A, B and C respectively on an Argand diagram.

The real number 1 is represented by the point D, and O is the origin.

(*d*) Show the points A, B, C and D on an Argand diagram. (2)

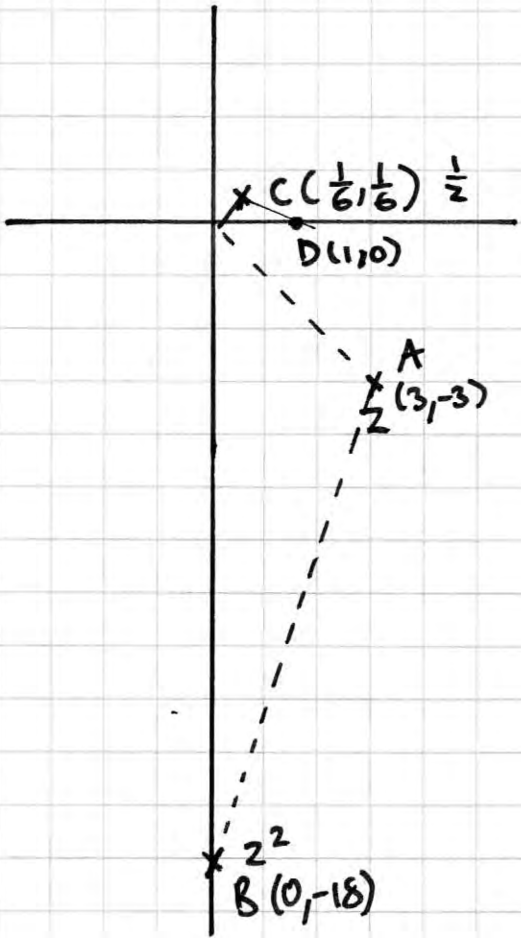
(*e*) Prove that $\triangle OAB$ is similar to $\triangle OCD$. (3)

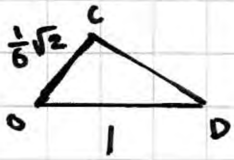
a) $Z^2 = (3-3i)(3-3i) = -18i$

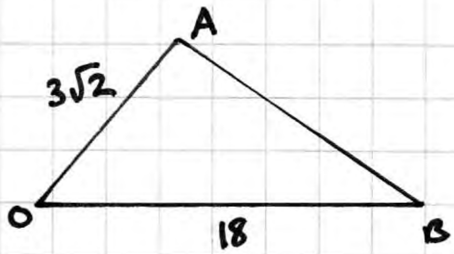
b) $\frac{1}{Z} = \frac{1}{3-3i} \times \frac{3+3i}{3+3i} = \frac{3+3i}{18} = \frac{1}{6} + \frac{1}{6}i$

c) $|z| = \sqrt{3^2+3^2} = 3\sqrt{2}$ $|z^2| = 18$

$\left|\frac{1}{z}\right| = \sqrt{\left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^2} = \frac{1}{6}\sqrt{2}$







$OC \times 18 = OA$

$OD \times 18 = OB$

$\therefore$ Mathematically Similar.

14. (*a*) Find the value of λ for which $\lambda x \cos 3x$ is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} + 9y = -12 \sin 3x. \qquad (4)$$

(*b*) Hence find the general solution of this differential equation. (4)

The particular solution of the differential equation for which $y = 1$ and $\frac{dy}{dx} = 2$ at $x = 0$, is $y = g(x)$.

(*c*) Find $g(x)$. (4)

(*d*) Sketch the graph of $y = g(x)$, $0 \le x \le \pi$. (2)

$y = \lambda x \cos 3x$

$y' = -3\lambda x \sin 3x + \lambda \cos 3x$

$y'' = -9\lambda x \cos 3x - 3\lambda \sin 3x - 3\lambda \sin 3x = -9\lambda x \cos 3x - 6\lambda \sin 3x$

$9y \quad 9\lambda x \cos 3x$ +

$+\underline{y''} = -9\lambda x \cos 3x - 6\lambda \sin 3x$

$-12 \sin 3x \equiv -6\lambda \sin 3x \quad \therefore \lambda = 2$

$\therefore y_{PI} = 2x \cos 3x$

b) $y = Ae^{mx}$ $\quad 9y + y'' = 0$

$y' = Ame^{mx}$ $\quad Ae^{mx}(m^2+9) = 0$

$y'' = Am^2e^{mx}$ $\quad \neq 0 \quad = 0 \quad m = \pm 3i$

$y_{CF} = A\cos 3x + B\sin 3x$

$\therefore y = (A + 2x)\cos 3x + B\sin 3x$

c) $x = 0, y = 1 \quad \underline{1 = A}$

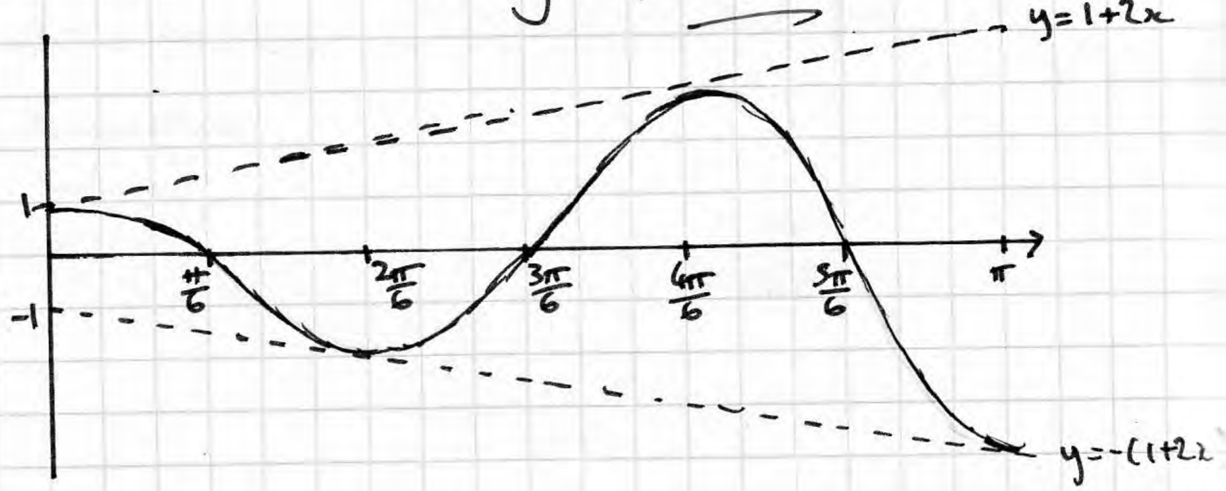
$y = (1+2x)\cos 3x + B\sin 3x$

$y' = -3(1+2x)\sin 3x + 2\cos 3x + 3B\cos 3x$

$x = 0 \; y' = 2 \quad 2 = 2 + 3B \quad \therefore B = 0$

$\therefore y = (1+2x)\cos 3x$

d)



15.

Figure 1

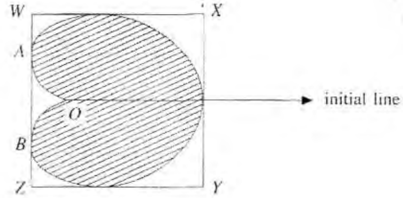


Figure 1 shows a sketch of the cardioid C with equation $r = a(1 + \cos\theta)$, $-\pi < \theta \leq \pi$. Also shown are the tangents to C that are parallel and perpendicular to the initial line. These tangents form a rectangle $WXYZ$.

(*a*) Find the area of the finite region, shaded in Fig. 1, bounded by the curve C. **(6)**

(*b*) Find the polar coordinates of the points A and B where WZ touches the curve C. **(5)**

(*c*) Hence find the length of WX. **(2)**

Given that the length of WZ is $\frac{3\sqrt{3}a}{2}$,

(*d*) find the area of the rectangle $WXYZ$. **(1)**

A heart-shape is modelled by the cardioid C, where a = 10 cm. The heart shape is cut from the rectangular card $WXYZ$, shown in Fig. 1.

(*e*) Find a numerical value for the area of card wasted in making this heart shape. **(2)**

$$\text{Area} = 2 \times \tfrac{1}{2}a^2\int_0^{\pi}(1+\cos\theta)^2 d\theta = a^2\int_0^{\pi} 1+2\cos\theta+(\tfrac{1}{2}+\tfrac{1}{2}\cos2\theta)d\theta$$

$$= a^2\int_0^{\pi} \tfrac{3}{2}+2\cos\theta+\tfrac{1}{2}\cos2\theta\, d\theta = \tfrac{1}{2}a^2\int_0^{\pi} 3+4\cos\theta+\cos2\theta\, d\theta$$

$$= \tfrac{1}{2}a^2\left[3\theta+4\sin\theta+\tfrac{1}{2}\sin2\theta\right]_0^{\pi} = \tfrac{1}{2}a^2(3\pi) = \frac{3\pi}{2}a^2$$

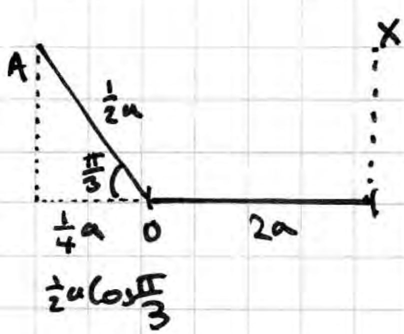
b) At A and B, tangent to curve perp to initial line $\theta = 0 \Rightarrow \frac{dx}{d\theta} = 0$

$$x = r\cos\theta = a(1+\cos\theta)\cos\theta = a(\cos\theta+\cos^2\theta)$$

$$\frac{dx}{d\theta} = a(-\sin\theta - 2\cos\theta\sin\theta) = 0 \Rightarrow 2\cancel{\sin\theta}\cos\theta = -\cancel{\sin\theta} \quad \therefore \cos\theta = -\tfrac{1}{2}$$

$$\therefore \theta = \tfrac{2\pi}{3}, -\tfrac{2\pi}{3} \qquad r = a(1+(-\tfrac{1}{2})) = \tfrac{1}{2}a \quad A(\tfrac{1}{2}a, \tfrac{2\pi}{3})\ B(\tfrac{1}{2}a, -\tfrac{2\pi}{3})$$

c)



$$\therefore WX = \tfrac{9}{4}a$$

d) $\text{Area} = \frac{9}{4}a \times \frac{3\sqrt{3}}{2}a = \frac{27\sqrt{3}}{8}a^2$

e) $\text{Waste} = \left(\frac{27\sqrt{3}}{8} - \frac{3\pi}{2}\right) \times 10^2 = \underline{113.3\text{cm}^2}$

16 A transformation T from the z-plane to the w-plane is defined by

$$w = \frac{z+1}{iz-1}, \quad z \neq -i,$$

where $z = x + iy$, $w = u + iv$ and x, y, u and v are real.

T transforms the circle $|z| = 1$ in the z-plane onto a straight line L in the w-plane.

(a) Find an equation of L giving your answer in terms of u and v. **(5 marks)**

(b) Show that T transforms the line $\operatorname{Im} z = 0$ in the z-plane onto a circle C in the w-plane, giving the centre and radius of this circle. **(6 marks)**

(c) On a single Argand diagram sketch L and C. **(3 marks)**

a) $wiz - w = z + 1$

$wiz - z = w + 1$

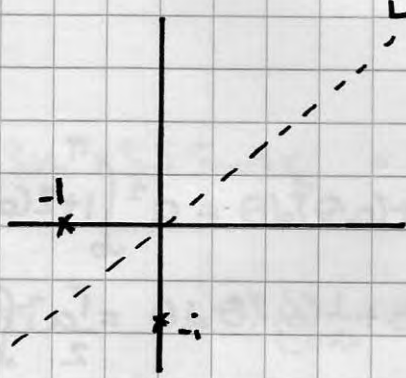
$z(wi - 1) = w + 1$

$|z||wi - 1| = |w + 1|$

$|z||wi^2 - i| = |w + 1|$

$|-w - i| = |w + 1|$

$|w + i| = |w + 1|$



$\therefore u = v$ in w-plane

b) Imaginary part of $z = 0$ $\therefore$ lies on the real axis.

$$z = \frac{w+1}{wi-1} = \frac{(u+1)+iv}{i(u+iv)-1} = \frac{(u+1)+iv}{-(v+1)+iu} \times \frac{[-(v+1)-iu]}{[-(v+1)-iu]}$$

$$z = \frac{-(u+1)(v+1) + uv + i(-u(u+1) - v(v+1))}{u^2 + (v+1)^2}$$

$\Rightarrow$ Imaginary part is zero $\Rightarrow$ $\dfrac{-u(u+1) - v(v+1)}{u^2 + (v+1)^2} = 0$

$\Rightarrow u(u+1) = -v(v+1) \Rightarrow u^2 + u + v^2 + v = 0$

$\Rightarrow \left(u + \frac{1}{2}\right)^2 + \left(v + \frac{1}{2}\right)^2 = \frac{1}{2}$ $\quad \therefore$ Circle $C\left(-\frac{1}{2}, -\frac{1}{2}\right)$ $r = \frac{1}{2}$